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ON AN ALTERNATIVE IN A SEVERAL-PERSON DIFFERENTIAL GAME

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A several-person position differential game is analyzed. Each player wants to lead the current position onto his own target set within his own constraints. A theorem, whose meaning follows, is proved under specified conditions. For each initial position either there is a single player resolving his own guidance problem under all possible counteractions of the remaining players or a control method can be found for all players such that not one of them can solve his own guidance problem if all the rest adhere to the control method mentioned. The proof is based on an assertion on an alternative for a two-person position differential game from [1]. Various aspects of several-person position differential games have been considered also in [2-4].

1. Basic concepts and definitions. Consider the system described by the equation

$$dx/dt = f(t, x, u_1, \ldots, u_k), \quad u_i \in P_i, \quad i = 1, 2, \ldots, k$$

$$f: R \times E^n \times E^{n_1} \times \ldots \times E^{n_k} \to E^n$$

Here $x \in E^n$ is the phase vector, u_i is the i-th player's control, P_i is a compactum in space E^{n_i}, the function f is continuous in all arguments. We use the following notation:

$$P = P_1 \times \ldots \times P_k, \quad P^{(i)} = P_1 \times \ldots \times P_{i-1} \times P_{i+1} \times \ldots$$
$$\ldots \times P_k$$
$$u = (u_1, \ldots, u_k), \quad u^{(i)} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k)$$
$$(u \in P, \; u^{(i)} \in P^{(i)})$$

By $(u_i^*, u^{(i)})$ we mean $(u_1, \ldots, u_i^*, \ldots, u_k)$. Everywhere the index i takes the values $1, 2, \ldots,$ (k is the number of players). For function f we additionally assume:

1) a constant $\varkappa > 0$ exits such that for any x

$$\| f(t, x, u) \| \leqslant \varkappa (1 + \| x \|)$$

uniformly in $t \in R$ and $u \in P$;

2) for each bounded set G from the space E^{n+1} of positions $\{t, x\}$ there exists

a positive constant λ_G such that

$$\| f(t, x^{(1)}, u) - f(t, x^{(2)}, u) \| \leqslant \lambda_G \| x^{(1)} - x^{(2)} \|$$

for all $\{t, x^{(1)}\} \in G$, $\{t, x^{(2)}\} \in G$ and $u \in P$;

3) for any vector $s \in E^n$, any position $\{t, x\} \in E^{n+1}$ and any number i

$$\min_{u^{(i)} \in P^{(i)}} \max_{u_i \in P_i} s' \cdot f(t, x, u_i, u^{(i)}) = \max_{u_i \in P_i} \min_{u^{(i)} \in P^{(i)}} s' \cdot f(t, x, u_i, u^{(i)})$$

In particular, functions of the form

$$f(t, x, u_1, \ldots, u_k) = f^{(1)}(t, x, u_1) + \ldots + f^{(k)}(t, x, u_k)$$

possess the last property.

Definition 1. A function $U_i : E^{n+1} \to P_i$ is called the i-th player's strategy.

Let Δ be a partitioning of the semi-axis $[t_*, \infty)$ by points τ_s, $s = 0, 1, 2, \ldots$, where $\tau_s < \tau_{s+1}$ for all s and $\tau_0 = t_*$.

Definition 2. Every absolutely continuous function satisfying the differential equations

$$dx_\Delta / dt = f(t, x_\Delta[t], U_{i_1}[\tau_s, x_\Delta[\tau_s]], \ldots, U_{i_l}[\tau_s, x_\Delta[\tau_s]]$$

$$u_{i_{l+1}}[t], \ldots, u_{i_k}[t]), \quad \tau_s \leqslant t \leqslant \tau_{s+1}, \quad s = 0, 1, 2, \ldots, \quad x_\Delta[t] = x_*$$

is called an Euler polygonal line $x_\Delta[t, t_*, x_*, U_{i_1}, \ldots, U_{i_l}]$, $t \in [t_*, \infty)$, issuing from the position $\{t_*, x_*\}$ and generated by the strategies $U_{i_1}, \ldots U_{i_l}$ of the l players $(1 \leqslant l \leqslant k)$. Here $u_{i_j}[\cdot]$, $j = l+1, \ldots, k$, are arbitrary Lebesgue-integrable functions for which $u_{i_j}[t] \in P_{i_j}$, $t \in [t^*, \infty)$.

Definition 3. Every function $x[\cdot]$ for which we can find in any interval $[t_*, \vartheta]$ a sequence of Euler polygonal lines $x_{\Delta^{(p)}}[t, t_*, x_*^{(p)}, U_{i_1}, \ldots, U_{i_l}]$, $t \in [t_*, \infty)$ $(p = 1, 2 \ldots)$ converging uniformly to it on this interval under the condition $\lim \sup_s (\tau_{s+1}^{(p)} - \tau_s^{(p)}) = 0$, as $p \to \infty$, is called a motion issuing from position $\{t_*, x_*\}$ and generated by the strategies $U_{i_1}, \ldots, U_{i_l}$ of the l players. The collection of all motions issuing from position $\{t_*, x_*\}$ and generated by the strategies $U_{i_1}, \ldots, U_{i_l}$ of the l players is called the sheaf of motions $X[t_*, x_*, U_{i_1}, \ldots, U_{i_l}]$.

Let $2k$ closed sets $M_1, \ldots, M_k, N_1, \ldots, N_k$ such that $M_i \subset N_i$, $i = 1, \ldots, k$, be given in the position space E^{n+1}.

Definition 4. The condition of contact with set M_i by the instant ϑ is fullfilled for the absolutely continuous function $x[t]$ $(t \in [t_*, \infty))$ if an instant $\tau \in [t_*, \vartheta]$ exists such that $x[\tau] \in M_i(\tau)$ and $x[t] \in N_i(t)$ for $t \in [t_*, \tau]$. Here and later on $A(t) = \{x \mid \{t, x\} \in A\}$ for any $A \subset E^{n+1}$.

Definition 5. Let A^ε be an open ε-neighborhood of set $A \subset E^{n+1}$ (for $A = \varnothing$ we assume $A^\varepsilon = \varnothing$). We say that the absolutely continuous function $x[t]$ $(t \in [t_*, \infty))$ evades set M_i^ε until the instant ϑ $(\vartheta \geqslant t_*)$ if $x[t] \,\overline{\in}\, M_i^\varepsilon(t)$ for any $t < \tau$. Here $\tau = \vartheta$ when $x[t] \in N_i^\varepsilon(t)$ for all $t \in [t_*, \vartheta]$ and $\tau = \min\{\tau^* \mid x[\tau^*] \,\overline{\in}\, N_i^\varepsilon(\tau^*)\}$ otherwise.

Let M and N be arbitrary closed sets from space E^{n+1}, and $M \subset N$. The two-person position differential game in which the i-th player solves the guidance problem (in the formulation given in [1]) by the instant ϑ onto set M within the phase constraints N and in which the unification of players remaining oppose him is denoted by the symbol $(i, (i), M, N, \vartheta)$. The symbol $((i), i, M, N, \vartheta)$ denotes the two-person differential game in which the guidance problem is solved by the totality of all players excepting the i-th one who now participates as a player-opponent. Let W_i be the maximal stable bridge in game $(i, (i), M_i, N_i, \vartheta)$. For brevity the set W_i is called the maximal i-stable bridge. In analogy with the notation adopted earlier, by the symbol U we mean the collection of k strategies $U_1, \ldots, U_k$ and by the symbol $U^{(i)}$, the collection of $k-1$ strategies $U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_k$.

2. Statement and proof of the main result.

Theorem. Let $W_i \cap W_j = \varnothing$ when $i \neq j$ for all $i, j = 1, 2, \ldots, k$. Then for every initial position $\{t_0, x_0\}$ either a number i and a strategy U_i^e of the i-player exist such that each motion from the sheaf $X[t_0, x_0, U_i^e]$ will satisfy the condition of contact with set M_i by the instant ϑ or $\alpha > 0$ and a collection $U^\circ = \{U_1^\circ, \ldots, U_k^\circ\}$ of strategies of the k players will be found such that for any i each motion from the sheaf $X[t_0, x_0, U^{(i)\circ}]$, where $U^{(i)\circ} = \{U_1^\circ, \ldots, U_{i-1}^\circ, U_{i+1}^\circ, \ldots, U_k^\circ\}$, will evade the α-neighborhood of set M_i until instant ϑ.

The theorem's proof is based on two lemmas. In the half-space $E_\vartheta^{n+1} = \{\{t, x\} \mid t \leqslant \vartheta\}$ let there be given the system of sets

$$S_1, \ldots, S_k, \quad D^{(1)}, \ldots, D^{(k)} \tag{2.1}$$

possessing the properties: 1) set $D^{(i)}$ is closed; 2) $S_i \cap S_j = \varnothing$ for $i \neq j$, $i, j = 1, \ldots, k$; 3) for all i the set $S^{(i)} = E_\vartheta^{n+1} \setminus S_i$ is a closed stable bridge in the game $((i), i, D^{(i)}, E_\vartheta^{n+1}, \vartheta)$. With the sets (2.1) introduced we associate a collection $U^\circ = \{U_1^\circ, \ldots, U_k^\circ\}$ of strategies of the k players, which we say is extremal to the set $S = E_\vartheta^{n+1} \setminus (S_1 \cup \ldots \cup S_k)$. The strategies in this collection are determined by the following conditions:

a) if $\{t, x\} \in S_i$ for some i and $S(t) = \varnothing$, then $U^{(i)\circ}(t, x)$ is found from the equality

$$\min_{U^{(i)} \in P^{(i)}} \max_{U_i \in P_i} s_*' \cdot f(t, x, u_i, u^{(i)}) = \max_{u_i \in P_i} s_*' \cdot f(t, x, u_i, u^{(i)\circ}), \quad s_* = x - w_*$$

where w_* is an arbitrary vector satisfying the equality

$$\| \{t, x\} - \{t, w_*\} \| = \min_{w \in S(t)} \| \{t, x\} - \{t, w\} \|$$

and $U_i^\circ(t, x)$ is assumed to be an arbitrary vector from P_i;

b) if $\{t, x\} \in S_i$ for some i but $S^{(i)}(t) = \varnothing$ or else $\{t, x\} \in S_i$, then $U_i^\circ(t, x)$ is an arbitrary vector $u \in P$.

Lemma 1. Let U° be the collection of strategies of the k players, extremal to set S. Then for any position $\{t_0, x_0\} \in S$ and any number i each motion from the sheaf $X[t_0, x_0, U^{(i)\circ}]$ goes onto set $D^{(i)}$ and does so before it leaves set $S^{(i)}$.

Proof. For any i the collection $U^{(i)\circ}$ of $k-1$ strategies is an extremal strategy to set $S^{(i)}$ in game $((i,) \, i, D^{(i)}, E_\vartheta^{n+1}, \vartheta)$. Therefore, by virtue of the stability of set $S^{(i)}$ all motions from the sheaf $X[t_*, x_*, U^{i\circ}]$, where $\{t_*, x_*\} \in S^{(i)}$, go onto set $D^{(i)}$ and do so before leaving set $S^{(i)}$. The lemma's validity now follows from the inclusion $\{t_0, x_0\} \in S$ and the equality $S = \cap S^{(i)}$.

By W and W_ε, where ε is an arbitrary positive number, we denote maximal stable bridges of the first player in some differential game of guidance by instant ϑ onto the target sets M and $\bar{M}^\varepsilon$ within the phase constraints N and $\bar{N}^\varepsilon$, respectively, (the overbar denotes the closure).

Lemma 2. If set M is bounded and set N has a bounded projection onto the time axis, then in any $\delta > 0$ we can find $\varepsilon > 0$ such that $W_\varepsilon \subset W^\delta$.

Proof. Indeed, assuming the contrary, we arrive at the existence of $\delta > 0$ and of a monotonically decreasing sequence $\{\varepsilon_n\}$ of positive numbers, converging to zero, such that $F_{\varepsilon_n} = W_{\varepsilon_n} \cap (E_\vartheta^{n+1} \setminus W^\delta) \neq \phi$ for all $n = 1, 2, \ldots$. Each set F_{ε_n}, $n = 1, 2, \ldots$, is a compactum and $F_{\varepsilon_{n+1}} \subset F_{\varepsilon_n}$ for all n; therefore, the set sequence $\{F_{\varepsilon_n}\}$ has a nonempty intersection F. Let $\{t_*, x_*\} \in F$. Then $\{t_*, x_*\} \overline{\in} W$. Hence, using the theorem on the alternative [1], we have that $\{t_*, x_*\} \in W_\varkappa$ for some $\varkappa > 0$. The latter contradicts the inclusion $\{t_*, x_*\} \in W_{\varepsilon_n}$ for all $n = 1, 2, \ldots$. This contradiction proves the lemma.

Proof of the theorem. If $\{t_0, x_0\} \in W_i$ for some i, then, obviously, the theorem's first assertion holds. Suppose that $\{t_0, x_0\} \overline{\in} W_i$ for any i. We show that the theorem's second assertion is valid. It suffices to consider the case when sets M_i and N_i are bounded. Indeed, we denote the maximal stable bridge in the game $(i, (i), M_i^*, N_i^*, \vartheta)$ by W_i^*. Here $M_i^* = M_i \cap B(t_0, x_0)$ and $N_i^* = N_i \cap B(t_0, x_0)$, and $B(t_0, x_0)$ is a closed sphere in E^{n+1} of a radius so large that all motions issuing from the position $\{t_0, x_0\}$ remain in this sphere during the interval $[t_0, \vartheta]$. It is obvious that $\{t_0, x_0\} \overline{\in} \cup W_i^*$. We now note that the theorem's second assertion for the position $\{t_0, x_0\}$ relative to sets M_i and N_i holds if and only if the theorem's second assertion for position $\{t_0, x_0\}$ is valid relative to sets M_i^* and N_i^*.

Let $W_{i\varkappa}$, $\varkappa > 0$, be the maximal stable bridge in the game $(i, (i), M_i^\varkappa, N_i^\varkappa, \vartheta)$. For each $\varepsilon > 0$ we consider the system of sets

$$S_{i\varepsilon} = \bigcup_{\varkappa \in [0, \varepsilon)} W_{i\varkappa}, \quad D_\varepsilon^{(i)} = (N_i^\varepsilon)^c \cup \{\{t, x\} \mid t = \vartheta\}$$

We prove that for sufficiently small $\varepsilon > 0$ it possesses all the properties of system (21) Indeed, the fulfillment of property 1) is obvious. Property 2) follows from Lemma 2. Let us show the fulfillment of property 3). By construction $S_{i\varepsilon}$ is the set of all positions for which, as for the initial positions, the i-th player's problem on contact with set $\bar{M}_i^\varkappa$ within set $\bar{N}_i^\varkappa$ by instant ϑ is solvable for at least one $\varkappa \in [0, \varepsilon)$. Consequently, $S_{i,\varepsilon}$ is the set of all positions for which, as for the initial positions, the i-th player's problem of evading the set $(N_i^\varepsilon)^c \cup \{\{t, x\} \mid t = \vartheta\}$ within set $(M_i^\varepsilon)^c$ by instant ϑ is solvable. Then it follows from the theorem on the alternative [1] that the set $E_\vartheta^{n+1} \setminus S_{i\varepsilon}$ is the maximal stable bridge in the game $((i), i, (N_i^\varepsilon)^c \cup \{\{t, x\} \mid t = \vartheta\}, (M_i^\varepsilon)^c, \vartheta)$. Consequently, this is the closed stable bridge for the game $((i), i, (N_i^\varepsilon)^c \cup \{\{t, x\} \mid t = \vartheta\}, E_\vartheta^{n+1}, \vartheta)$, which signifies the fulfillment of property 3).

Applying Lemma 1, we obtain that if $\{t_0, x_0\} \in S_\varepsilon = E_\vartheta^{n+1} \setminus (S_{1\varepsilon} \cup \cdots \cup S_{k\varepsilon})$ and $\varepsilon > 0$ is sufficiently small, then for any i each motion in the sheaf $X[t_0, x_0, U^{(i)\circ}]$, where U° is the collection of strategies extremal to set S_ε, goes onto set $(N_i^\varepsilon) \cup \{\{t, x\} \mid t = \vartheta\}$ before leaving the set $S_\varepsilon^{(i)} = E_\vartheta^{n+1} \setminus S_{i\varepsilon}$. By virtue of the obvious inclusion $M_i^\varepsilon \subset S_{i\varepsilon}$ each motion from the sheaf $X[t_0, x_0, U^{(i)c}]$ evades the ε-neighborhood of set M_i. From Lemma 2 follows the existence of $\varkappa > 0$ so small that $\{t_0, x_0\} \in S_\varepsilon$ for all $\varepsilon \leqslant \varkappa$. This completes the theorem's proof.

Notes. 1°. Let $k = 2$ and $N_2 = \phi$. Then the theorem presented is equivalent to the theorem on the alternative for a two-person differential game in the formulation from [1] if we set $N = N_1$ and $M = M_1$.

Indeed, let us denote the theorem from [1] by the symbol A and the theorem proved here by symbol B. Let us fix an initial position. From $N_2 = \phi$ follows $W_2 = \phi$. Therefore, the first possibility of Theorem A holds if and only if the first possibility of Theorem B holds. Suppose that the second possibility of Theorem A is realized for the initial position selected, i. e. the second player's evasion strategy V° is found. Then the collection of strategies U_1° and U_2° from Theorem B can be constructed by setting $U_2 = V^\circ$ and taking an arbitrary U_1°. Conversely, if the above-mentioned collection of strategies U_1° and U_2° exists, then the second player's evasion strategy V° can be chosen from the condition $V^\circ = U_2^\circ$.

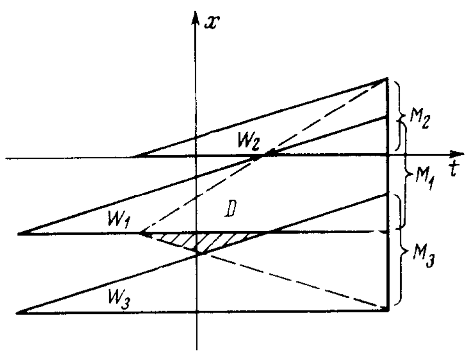

Fig. 1.

2°. As we see from its definition, for the actual construction of the strategy collection U° we can make use of the results in [1] for two-person differential games.

3°. The requirement in the theorem's hypothesis that the maximal i-stable bridges are nonintersecting is an essential one. This can be seen from the following example. Consider the three-person differential game

$$x^{\cdot} = u_1 + u_2 + u_3 \tag{2.2}$$
$$u_1 \in [-0.1,\ 0.2],\quad u_2 \in [-0.2,\ 0.1],\quad u_3 \in [0,\ 0.3],\quad \vartheta = 5$$
$$M_1 = \{\{t, x\} \mid t = 5,\ x \in [-2,1]\},\quad M_2 = \{\{t, x\} \mid t = 5,\ x \in [0,\ 2]\}$$
$$M_3 = \{\{t, x\} \mid t = 5, x \in [-4,\ -1]\},\quad N_1 = N_2 = N_3 = E^2$$

The sets M_i, $i = 1, 2, 3$, and the maximal i-stable bridges corresponding to them are shown in Fig. 1. Let D be the set of all positions $\{t, x\}$ in the half-space E_ϑ^2 such that any motion of system (2.2), issuing from $\{t, x\}$, hits the set $M_1 \cup M_2 \cup M_3$ (the set is shown in the Fig. 1 by dashed lines). We set $S = W \cap D$, where $W = E_\vartheta^2 \setminus (W_1 \cup W_2 \cup W_3)$. We can verify that $S \neq \varnothing$. The theorem's statement is incorrect for any position $\{t_0, x_0\} \in S$.

The result obtained in this paper can prove useful in the investigation of several-person differential games of kind, wherein a payoff for one of the players involves a loss for the rest, and in cases when no player has a payoff and each participant retains the right to prolong the conflict under certain altered conditions.

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